Floating Reference Frames for Flexible Spacecraft

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Floating reference frames which move with the flexible body under dynamic analysis offer the advantages of a linear vibration analysis in the presence of large sytem rotations. When the deformations of an elastic continuum are expanded in terms of the free-free modes on an unconstrained system, the rigid-body modes are found to be fixed in a reference frame called the Tisserand frame, with respect to which the relative momentum is zero. This result also guarantees the independence of small variations of frame motions and coordinates for all modes with nonzero natural frequencies, a condition which can greatly simplify the formulation of equations of motion. A modified Tisserand constraint is introduced in order to define a floating reference frame with similar properties for an elastic body which contains spinning rotors.

Introduction

FLOATING reference frames have long been used in spacecraft dynamics. The most popular has been the locally attached or body-fixed frame. This is a very attractive choice when a central rigid body is readily identifiable. However, the next generation of spacecraft will be characterized by distributed flexibility, thus making the choice of a locally attached reference frame arbitrary. In this paper, a review is made of the more general ways of defining a floating reference frame using constraint relationships. Constraint equations must accompany the introduction of an unattached floating reference frame because this implies the introduction of six redundant variables, the behavior of which cannot be defined by the equations of dynamics alone. The advantage is that some unattached frames can be shown to follow the body in an "optimal" manner, as recently and most succinctly shown by de Veubeke.[1] Earlier work in the subject of floating reference frames can be found in Refs. 2-5.

This paper concentrates on the treatment of the constraint relationships for the Tisserand frame. The choice of mode shapes to be used for expansion of the deformations relative to the floating frame is crucial. With an unwise choice of mode shapes, the constraint relationship may depend on all of the coordinates and may be difficult to deal with. The simplest constraint relationship results from the use of the free-free modes of an unconstrained system. The orthogonality of the rigid body modes (of zero natural frequency) to the deformational modes (of nonzero natural frequency) allows for a rather elegant a priori evaluation of the constraint relationships. The constraint relationships then involve only the rigid-body modal coordinates, which are constrained to be zero. This result may then be interpreted as a requirement that the rigid-body mode is fixed relative to the Tisserand frame. The remaining coordinates are then independent of the Tisserand frame coordinates and can be chosen so as to be independent of each other, a condition which can greatly simplify the formulation of equations of motion. This special case of the Tisserand constraint that uses the nonzero-frequency free-free modes of an unconstrained system is termed the "mode shape constraint," because each deformational mode shape contributes zero net momentum and automatically satisfies the constraint. An example involving a two-bar linkage is worked out in order to demonstrate the properties of the mode shape constraint.

The classical Tisserand constraint cannot be employed for systems with rotating internal members. This class of systems represents flexible spacecraft with momentum exchange controllers, such as reaction wheel, momentum wheel, and control moment gyro control systems. Several alternate extensions to the Tisserand frame for such systems are discussed, and a modified Tisserand constraint is introduced. The properties of this constraint relationship are discussed, and the applicability of the mode shape constraint is demonstrated.

Floating Frames for Deformable Bodies

For the problems discussed here, we are concerned with small deformations of elastic bodies. Specifically, we will define a deformable body to be a body for which the relative displacements are so small that only first-order terms need be retained in the analysis. The body is allowed overall motion that is completely unrestricted; that is, the body has complete freedom of motion in responding to impressed moments and forces.

For such a body, it is difficult to specify a set of axes from which to measure deformations. If an inertially fixed set of axes is chosen, the displacements relative to these axes may grow large if the body undergoes any appreciable rotation due to an externally applied moment. Then the dynamic analysis of the system using these displacements would require more than a first-order analysis. In order to simplify the analysis, the idea presents itself to use a frame that somehow moves with the body. If a frame moves with the body, or "floats," in the proper way, then the displacements measured relative to this floating frame will be small. The dynamic analysis of the system may then be pursued using a first-order vibration analysis. It is the first-order-analysis property that makes a floating frame attractive.

Five types of floating frames will be treated here: 1) locally attached frame, 2) principal axis frame, 3) Tisserand frame, 4) Buckens frame, and 5) rigid-body mode frame. Before these different types of frames are covered in detail, several general characteristics of floating frames will be discussed. A deformable body experiences relative displacement, and therefore the system inertia quantities do not remain constant. In the formulation of equations of motion, the system angular momentum and kinetic energy are functions of the relative

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displacements and possess a more complicated structure than for rigid bodies. The system angular momentum for a deformable body about its center of mass is written as

$$H = \int_D \rho \times \dot{\rho} \, dm \tag{1}$$

where

- ρ = vector from the center of mass to a generic mass element
- $\dot{\rho}$ = $({}^i d/dt)\rho$ = time derivative of ρ relative to inertial space
- dm = generic mass element
- D = integration over the deformable body
- H = angular momentum of the system about the center of mass

If a floating frame f is introduced with its origin at the center of mass, then

$$\dot{\rho} = \mathring{\rho} + \omega^{fi} \times \rho \tag{2}$$

where

- $\mathring{\rho}$ = $({}^f d/dt)\rho$ = time derivative of ρ relative to floating frame f
- ω^{fi} = angular velocity of frame f relative to inertial space

Using this relationship to evaluate the angular momentum yields

$$H = \mathbf{I} \cdot \omega^{fi} + \int_D \rho \times \mathring{\rho} \, dm \tag{3}$$

since

$$\int_D \rho \times (\omega^{fi} \times \rho) \, dm = \mathbf{I} \cdot \omega^{fi} \tag{4}$$

where $\mathbf{I}$ is the inertia dyadic about the mass center. The second quantity in Eq. (3) is the angular momentum relative to the floating frame. This is referred to as the internal angular momentum. The first term of Eq. (3) is structurally identical to the rigid-body angular momentum, but since the body is deformable, the inertia dyadic is not constant. It should be noted that for small (first-order) displacements relative to the floating frame, the variations in the inertia dyadic will be first- and higher-order terms.

The kinetic energy of a body with an inertially fixed center of mass may be written

$$T = \frac{1}{2} \int_D \dot{\rho} \cdot \dot{\rho} \, dm \tag{5}$$

Using the relationship for the time derivative in Eq. (2) allows the kinetic energy to be rewritten using a floating frame as

$$T = \frac{1}{2} \omega^{fi} \cdot \mathbf{I} \cdot \omega^{fi} + \omega^{fi} \cdot \int_D (\rho \times \mathring{\rho}) dm + \frac{1}{2} \int_D \mathring{\rho} \cdot \mathring{\rho} dm \tag{6}$$

The first term of this equation has the same structural form as the kinetic energy of a rigid body. Again, this quantity differs from the rigid-body quantity because the inertia dyadic is not constant. The second term includes the same internal angular momentum expression found earlier in Eq. (3). The last term is the internal kinetic energy, since it represents kinetic energy contributed by the dot product of velocities relative to the frame.

We will now deal with the specific definitions for the most commonly available floating frames. The first type to be discussed is the locally attached frame. For this frame, a subbody or mass element is identified in the deformable body, and a frame is defined that follows the motion of this subbody or mass element. An example of this type of frame may be associated with a spacecraft with a rigid central body and a flexible appendage; a reference frame may be attached to the central rigid body. The angular momentum and kinetic energy for a locally attached frame would be given by Eqs. (3) and (6), respectively. As a general rule, no simplification of these expressions could be guaranteed for the locally attached reference frame. Of course, this choice involves no redundant variables, and hence no constraint relationships.

The next type of floating reference, the principal-axis frame, does offer some simplification of the expressions for angular momentum and kinetic energy. One could imbed in this frame a set of axes for which the origin of the axes would be the center of mass, and the orientation of the axes would be such that the corresponding inertia matrix of the deformable body would be diagonal. Thus, these axes would define the location of the principal axes of inertia. The components of a diagonal inertia matrix are called the moments of inertia for principal axes, and the products of inertia are all zero. Because the inertia matrix contains only three components, the calculations for angular momentum and kinetic energy are simplified. However, this is done at the expense of introducing three constraint relationships that require the products of inertia all to be zero. The constraint relationships are

$$I_{12} = I_{21} = -\int_D \rho_1 \rho_2 dm = 0 \tag{7a}$$

$$I_{13} = I_{31} = -\int_D \rho_1 \rho_3 dm = 0 \tag{7b}$$

$$I_{23} = I_{32} = -\int_D \rho_2 \rho_3 dm = 0 \tag{7c}$$

where (ρ_1, ρ_2, ρ_3) are the components of ρ in the floating frame.

For the Tisserand or mean axes frame, the expressions for angular momentum and kinetic energy are structurally simplified by moving the axes so as to set the internal angular momentum always to zero. The requirement is also made that the internal linear momentum be zero. This latter constraint is accomplished by the simple requirement that the origin of the frame be located at the center of mass. In order to set the internal angular momentum to zero, a constraint relationship is introduced

$$\int_D \rho \times \mathring{\rho} \, dm = 0 \tag{8}$$

From Eq. (3), the angular momentum is thus

$$H = \mathbf{I} \cdot \omega^{fi} \tag{9}$$

This is structurally identical to the rigid-body form, although, as noted, the inertia matrix is not a constant for a deformable body as it would be for a rigid body relative to axes fixed in that body.

The Tisserand frame provides the minimum kinetic energy relative to the floating frame, since

$$\delta\left(\frac{1}{2} \int_D \mathring{\rho} \cdot \mathring{\rho} dm\right) = \int_D \mathring{\rho} \cdot \delta\mathring{\rho} \, dm = 0 \tag{10}$$

This will be evaluated for all changes in relative velocity due to frame motion, but the position of the axes will not be perturbed at the epoch of comparison. We represent the position of a mass element by

$$r = R + \rho \tag{11}$$

where R is the vector from an inertially fixed reference point to the center of mass. The absolute velocity may then be written using Eq. (2).

$$\dot{r} = \dot{R} + \omega^{fi} \times \rho + \overset{\circ}{\rho} \tag{12}$$

It may then be noted that the absolute velocity does not depend on the frame motion.

$$\delta\dot{r} = \delta\dot{R} + \delta\omega^{fi} \times \rho + \delta\overset{\circ}{\rho} = 0 \tag{13}$$

Solving for $\delta\overset{\circ}{\rho}$, the minimization equation may be evaluated.

$$\int_D \overset{\circ}{\rho} \mathrm{d}m \cdot \delta\dot{R} + \int_D \rho \times \overset{\circ}{\rho}\ \mathrm{d}m \cdot \delta\omega^{fi} = 0 \tag{14}$$

Since $\delta\dot{R}$ and $\delta\omega^{fi}$ may be chosen independently, this gives the constraint relationships setting the relative momentum to zero

$$\int_D \overset{\circ}{\rho}\ \mathrm{d}m = 0 \qquad \int_D \rho \times \overset{\circ}{\rho}\ \mathrm{d}m = 0 \tag{15}$$

This development is due to de Veubeke.[1]

Going back to the constraint relationship, let us evaluate this in greater detail. First we introduce a new expression for the position relative to the center of mass

$$\rho = \bar{\rho} + u \tag{16}$$

where $\bar{\rho}$ is the position of a generic mass element in the underofmed state. This vector ρ is fixed in the floating reference frame and may be thought of as the station location of a mass element. The vector u represents the deformation of a generic mass element. For a deformable body, this will be a first-order quantity. The derivative of the position of a generic mass element relative to the center of mass is then written as

$$\overset{\circ}{\rho} = \overset{\circ}{\bar{\rho}} = \overset{\circ}{u} = \overset{\circ}{u} \tag{17}$$

since $\bar{\rho}$ is fixed in the frame and has no derivative relative to the frame. The constraint relationship is then

$$\int_D \rho \times \overset{\circ}{u}\ \mathrm{d}m = 0 \tag{18}$$

The next step is our examination of the constraint relationship is to introduce a separation of variables for the deformations. This separation of variables may be written in vector form as

$$u = \sum_{i=1}^{n} \phi^i \eta_i \tag{19}$$

where the mode shape ϕ^i depends only on the spatial variables, and η_i is the time-dependent modal coordinate. Using the separation of variables in the constraint relationship yields

$$\sum_{j=1}^{n} \int_D \left[\left(\bar{\rho} + \sum_{i=1}^{n} \phi^i \eta_i\right) \times \phi^j\right] \mathrm{d}m\dot{\eta}_i = 0 \tag{20}$$

This is a vector constraint of the Pfaffian form

$$\sum_{j=1}^{n} a_j(\eta)\dot{\eta}_j = 0 \tag{21}$$

where

$$a_j(\eta) = \int_D \left[\left(\bar{\rho} + \sum_{i=1}^{n} \phi^i \eta_i\right) \times \phi^j\right] \mathrm{d}m \tag{22}$$

If the constraint relationship [Eq. (20)] possesses an integral, then the constraint is holonomic. It is important to note that if a Tisserand constraint is used in a Lagrangian formulation, then the equations of motion must include Lagrange multipliers for the general case. The Tisserand frame is a general concept and may be applied to a system where the deformations are large. In the case of a deformable body as previously defined, the relative displacements are small and may be treated analytically as first-order quantities. The analysis may then proceed using first-order quantities and may ignore second- and higher-order terms. It is proceeding in this direction which led to the introduction of the Buckens frame.[2]

The Buckens constraint relationship is holonomic and may be written as

$$\int_D \bar{\rho} \times u\ \mathrm{d}m = 0 \tag{23}$$

Again the origin of the system is placed at the center of mass. The derivative relative to the frame of the Buckens constraint is identical to the Tisserand constraint if this relationship holds

$$\int_D u \times \overset{\circ}{u}\ \mathrm{d}m = 0 \tag{24}$$

Since second-order quantities may be ignored for a deformable body, Eqs. (24) and (18) are in this limited context equivalent. The Buckens frame, however, differs formally from the Tisserand frame in that it minimizes the squares of the relative displacements rather than minimizing the relative kinetic energy,[1] that is, for the Buckens frame

$$\delta\left(\frac{1}{2}\int_D u \cdot u\ \mathrm{d}m\right) = \int_D u \cdot \delta u\ \mathrm{d}m = 0 \tag{25}$$

Equation (11) may be represented in matrix form by

$$\{\rho\} = [C](\{r\} - \{R\}) \tag{26}$$

where $[C]$ is the direction-cosine matrix from an inertial to the Buckens frame, $\{\rho\}$ consists of the components in the Buckens frame, and $\{r\}$ and $\{R\}$ are components in an inertial frame. We shall now evaluate δu in the minimizing equation:

$$\{\delta u\} = \{\delta\rho\} = [\delta C]\ (\{r\} - \{R\}) - [C]\{\delta R\} + [C]\{\delta r\} \tag{27}$$

It is realized that the absolute virtual displacement $\{\delta r\}$ is zero, so that the substitutions

$$\{r\} - \{R\} = [C]^T\{\rho\} \qquad [\tilde{\delta\omega}] = -[\delta C][C]^T \tag{28}$$

will yield the expression for the virtual relative displacement

$$\{\delta u\} = -[\tilde{\delta\omega}]\{\rho\} - [C]\{\delta R\} \tag{29}$$

The minimizing equation then becomes

$$\{\delta\omega\}^T \int_D [\tilde{\rho}]\{u\}\mathrm{d}m + \{\delta R\}^T [C]^T \int_D \{u\}\ \mathrm{d}m = 0 \tag{30}$$

where the triple scalar product has been rearranged. By recognizing that $[\tilde{u}]\ \{u\} = 0$, and that $\{\delta\omega\}$ and $\{\delta R\}$ are arbitrary, we have the Buckens constraint relationships[1]:

$$\int_D u\mathrm{d}m = 0 \qquad \int_D \bar{\rho} \times u\ \mathrm{d}m = 0 \tag{31}$$

Since the Buckens and linearized Tisserand constraints are equivalent for a deformable body as defined earlier, they will both be referred to here as the Tisserand constraint. This choice is made because of the historical precedence of the Tisserand constraint, and the fact that it is more common in the literature. This is not to minimize the contribution made by Buckens, for it is the first-order form which will allow the greater use of the Tisserand frame that is explored later.

The last frame to be described is the rigid-body mode frame. This concept arises in structural dynamics for semidefinite systems. A semidefinite system is one for which the strain energy may be zero without the motion being zero. Unrestrained systems, or systems without supports, are typical examples of semidefinite systems. A rigid-body mode is defined as a displacement which results in zero strain energy. The rigid-body mode frame follows the displacement which results in zero strain energy. There is an essential difference between the rigid-body mode and the equilibrium position of a body. The rigid-body mode is associated with zero strain energy, a condition which describes the equilibrium position of a body only if it is at rest in inertial space. If a body is spinning at a given rate, the equilibrium position will not coincide with the rigid-body mode, since the "centrifugal forces" may be thought of as inducing a nonzero strain energy at equilibrium.

The constraint relationship associated with the rigid-body mode is simply that the strain energy be zero. The difficulty of working with this constraint is circumvented by showing a relationship between the Tisserand frame for deformable bodies and the rigid-body mode frame. This allows the rigid-body mode frame to be easily applied. This is pursued in the next section.

In order to give insight into the concept of a floating reference frame, a simple example problem has been formulated. This example system is shown in Fig. 1 and consists of two uniform rigid bodies that are connected by a line hinge. All motion is planar. In the undeformed position, the two bodies are aligned in a straight line. The length of body 1 is l, and the length of body 2 is $l/\sqrt{2}$. The mass per unit length of body 2 is twice that of body 1. This places the center of mass for the undeformed body at the hinge point. The coordinate systems for the example are shown in Fig. 2. The deformation of the system is specified by the angle α between the two bodies. A set of axes, with respect to which unit vectors $\hat{b}_1$, $\hat{b}_2$, and $\hat{b}_3$ are fixed, is fixed in body 2 with its origin at the hinge point. The location of the hinge point relative to the center of mass is specified by the vector $\boldsymbol{R}$. A floating fame, in which unit vectors $\hat{f}_1$, $\hat{f}_2$, and $\hat{f}_3$ are fixed, has an imbedded point fixed at the center of mass and an imbedded line which makes an angle β with the body-fixed axes. A mass element is located by a vector $\boldsymbol{x}$ relative to the hinge point with the subscript specifying the associated body. The center of mass is defined by

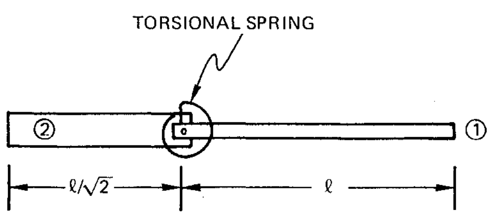

Fig. 1 Example system.

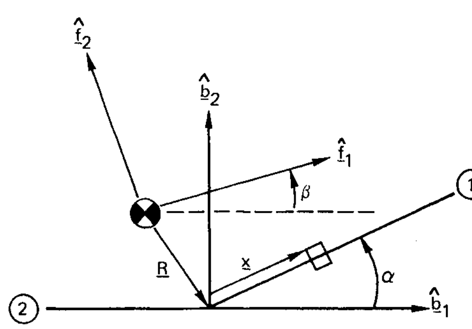

Fig. 2 Coordinate systems.

$$\int_D \boldsymbol{\rho}\, dm = 0 \tag{32a}$$

This becomes for the example

$$\int_0^l (\boldsymbol{R}+\boldsymbol{x}_1)\, m dx + \int_0^{l/\sqrt{2}} (\boldsymbol{R}+\boldsymbol{x}_2) 2m dx = 0 \tag{32b}$$

where

$$\boldsymbol{x}_1 = x(\cos\alpha \hat{\boldsymbol{b}}_1 + \sin\alpha \hat{\boldsymbol{b}}_2) \qquad \boldsymbol{x}_2 = -x\hat{\boldsymbol{b}}_1 \tag{32c}$$

This yields the position vector of the hinge point relative to the center of mass

$$\boldsymbol{R} = \hat{\boldsymbol{b}}_1 \left[\frac{l}{4}(1-\cos\alpha)\right] - \hat{\boldsymbol{b}}_2 \left(\frac{l}{4}\sin\alpha\right) \tag{32d}$$

Note that the center of mass is at the hinge point for zero deformations.

The Tisserand frame will have its origin at the center of mass and will obey the constraint relationship

$$\int_D \boldsymbol{\rho} \times \mathring{\boldsymbol{\rho}}\, dm = 0 \tag{32e}$$

where

$$\boldsymbol{\rho}_1 = \boldsymbol{R} + \boldsymbol{x}_1 \qquad \boldsymbol{\rho}_2 = \boldsymbol{R} + \boldsymbol{x}_2 \tag{32f}$$

The constraint then becomes

$$\int_0^l m(\rho_{1_x}\dot{\rho}_{1_y} - \rho_{1_y}\dot{\rho}_{1_x})\, dx' + \int_0^{l/\sqrt{2}} 2m(\rho_{2_x}\dot{\rho}_{2_y} - \rho_{2_y}\dot{\rho}_{2_x})\, dx = 0 \tag{32g}$$

Evaluating the vectors in Eq. (32f) in the floating reference frame so as to facilitate the derivatives involved yields

$$\boldsymbol{\rho}_1 = \{\hat{f}\}^T \begin{Bmatrix} (l/4)\cos\beta + (x - l/4)\cos(\alpha-\beta) \\ -(l/4)\sin\beta + (x-l/4)\sin(\alpha-\beta) \\ 0 \end{Bmatrix} \tag{32h}$$

$$\boldsymbol{\rho}_2 = \{\hat{f}\}^T \begin{Bmatrix} -(x - l/4)\cos\beta - (l/4)\cos(\alpha-\beta) \\ (x-l/4)\sin\beta - (l/4)\sin(\alpha-\beta) \\ 0 \end{Bmatrix} \tag{32i}$$

where $\{\hat{f}\}$ is the 3×1 array of vectors $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$. With these relationships and the derivatives of the components, the constraint relationship may be evaluated. After the necessary algebra and integrations, Eq. (32g) becomes

$$(0.2342 + 0.0991\cos\alpha)\dot{\alpha} - (0.3707 + 0.1982\cos\alpha)\dot{\beta} = 0 \tag{32j}$$

This constraint is in the Pfaffian form, and since it possesses an integral, the Tisserand constraint here is holonomic. If the deformations are small, then the constraint shows that the preceding frame must move relative to the bodies in a specific manner, that is

$$\dot{\beta}=0.586\dot{\alpha} \tag{32k}$$

One special case of the rotation of the Tisserand frame relative to inertial space should be mentioned. For a moment-free body with zero initial angular momentum, the angular momentum will remain zero due to conservation laws. For this case, by reference to Eq. (9), the frame angular velocity must be zero since the inertia matrix is positive definite. Thus, the Tisserand frame in this special case will be inertially fixed, while the principal-axis frame may move.

The principal-axis and Tisserand frames both had applications to problems in the physical sciences during the late nineteenth century. The hypothesis was presented that the glacial periods were a result of the movement of the axis of rotation relative to the surface of the Earth. The use of floating frames was introduced in order to formulate equations of motion for a body undergoing small deformations due to elevation and subsidence of continents and sea beds. The researches of George Darwin were the most detailed in the evaluation of specific results.[6] He introduced a principal-axis frame and derived equations of motion for this frame. He was able to show that a movement of the poles of 8 deg would require one half of the Earth's surface to be deflected by nearly 2 miles. This would effectively make continents out of oceans and vice versa, and he stated that some other explanation would have to be found for the glacial periods.

The mean axes which were popularized by Tisserand and which bear his name were first introduced by Glyden in his study of the rotation of the Earth.[7,8] The work of Glyden uses this frame to derive an expression for the deviation of the mean axes from the principal axes of the system. The application of the floating reference frame was similar in intent to that of Darwin, but his conclusions regarding the motion of the poles were more analytical in nature and not as concrete.

Mode Shape Constraint

There were two requirements made in order to define the Tisserand frame for a deformable body. The first was that the center of mass be fixed in the frame. This would then insure that the system possessed no linear momentum relative to the frame. But, by introducing this constraint, three additional variables were required to specify the location of the point of the frame occupied by the mass center, which point may be referred to as the frame origin. The second defining requirement was that the system have no angular momentum relative to the frame. This constraint then introduced an additional three coordinates to specify the angular orientation of the frame. The net result of injecting the Tisserand frame into the problem for a deformable body is to expand the dimension of the problem by six and to interrelate the coordinates by six scalar equations (two vector equations). It is the interrelation of the coordinates, and the resulting fact that they are no longer linearly independent, which can prove troublesome in the formulation of the equations of motion. Constraint relationships require the use of Lagrange multipliers when Lagrange's equations are used. It would be extremely advantageous to evaluate the constraint relationships and reduce the order of the system before the equations of motion are formulated.

There exists a special relationship between the linearized Tisserand frame and the free-free modes of an unconstrained system which we shall call the mode shape constraint. This relationship employs the orthogonality conditions that exist between the rigid-body modes and the deformational modes (which have nonzero natural frequencies) to allow a rather elegant a priori evaluation of the Tisserand constraint. This is accomplished by setting all rigid-body modal coordinates to zero. This effectively reduces the order of the system to the original value, and the remaining coordinates are all independent.

The constraint relationships that define a linearized Tisserand frame for a deformable body are given by

$$\int_D \rho \mathrm{d}m=0 \qquad \int_D \bar{\rho}\times \mathring{u}\ \mathrm{d}m=0 \tag{33}$$

A separation of variables is introduced using the free-free mode shapes of an unconstrained system

$$u=\sum_{j=0}^{n}\phi^j\eta_j \tag{34}$$

where the modal shapes for distinct eigenvalues are orthogonal

$$\int_D \phi^i\cdot\phi^j \mathrm{d}m=0 \quad (i\neq j) \tag{35}$$

The constraint relationships are then written as

$$\sum_{i=0}^{n}\int_D \phi^j \mathrm{d}m\eta_j=0 \tag{36}$$

and

$$\sum_{j=0}^{n}\int_D \bar{\rho}\times\phi^j \mathrm{d}m\dot{\eta}_j=0 \tag{37}$$

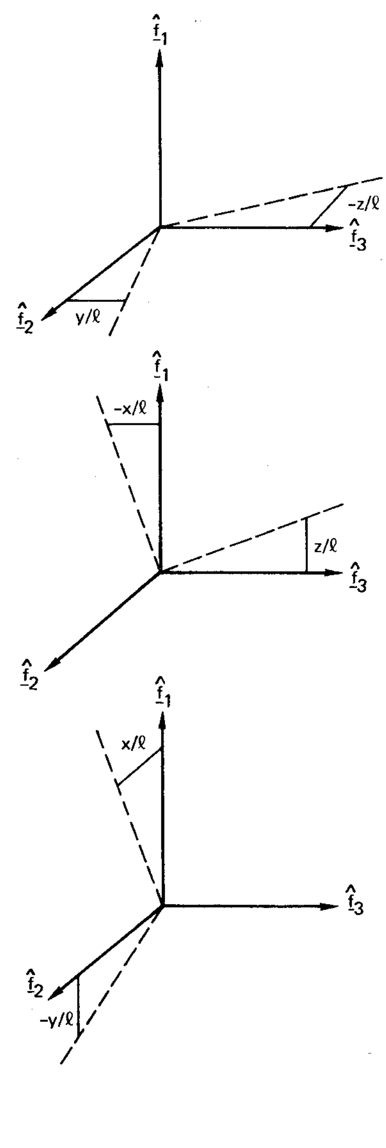

Fig. 3 Rotational rigid-body modes.

In the foregoing, the zero subscript has been used to refer to all of the rigid-body modes. These consist of a total of six modes. The three translational rigid-body modes are taken as uniform translation along each of the axes. They may be written as

$$\phi_i^0 = c\hat{f}_i \quad (i=1,2,3) \tag{38}$$

where c is a constant. The three rotational rigid-body modes are taken as small rotations about each of the axes. They are pictured in Fig. 3 and are written as

$$\phi_4^0 = (y/l)\hat{f}_3 - (z/l)\hat{f}_2 \tag{39a}$$

$$\phi_5^0 = (z/l)\hat{f}_1 - (x/l)\hat{f}_3 \tag{39b}$$

$$\phi_6^0 = (x/l)\hat{f}_2 - (y/l)\hat{f}_1 \tag{39c}$$

The translational rigid-body modes are orthogonal to the rotational rigid-body modes since the undeformed position has its origin at the center of the mass and

$$\int_D \bar{\rho}\,\mathrm{d}m = 0 \tag{40}$$

It will be most fruitful in the evaluation of the constraint relationships to investigate the orthogonality properties of the free-free modes further. Let ϕ^j be a deformational mode corresponding to a nonzero natural frequency:

$$\phi^j = \{\hat{f}\}^T \begin{Bmatrix} \phi_1^j \\ \phi_2^j \\ \phi_3^j \end{Bmatrix} \tag{41}$$

This mode will be orthogonal to the translational modes, yielding the relationships

$$\int_D \phi_i^0 \cdot \phi_j \mathrm{d}m = c\int_D \phi_i^j \mathrm{d}m = 0 \quad (i=1,2,3) \tag{42}$$

This may be written in the vector form as

$$\int_D \phi^j \mathrm{d}m = 0 \tag{43}$$

This is recognized as the coefficient in the center-of-mass constraint of η. Coupled with the orthogonality of the rigid-body translational modes to the rotational modes shown in Eq. (40), the center-of-mass constraint can be rewritten as a relationship involving only the translational rigid body modal coordinates

$$\int_D \phi_1^0 \mathrm{d}m\eta_{01} + \int_D \phi_2^0 \mathrm{d}m\eta_{02} + \int_D \phi_3^0 \mathrm{d}m\eta_{03} = 0 \tag{44}$$

Evaluating the integrals will yield the three scalar equations

$$M\eta_{0i} = 0 \quad (i=1,2,3) \tag{45}$$

where M is the total system mass. It is now quite clear that the center-of-mass constraint requires that the translational rigid-body modal coordinates be zero.

Now we shall turn to the constraint setting the internal angular momentum to zero by studying the orthogonality of a deformational mode ϕ^j to the rotational rigid body modes. The following orthogonality condition holds

$$\int_D \phi_i^0 \cdot \phi^j \mathrm{d}m = 0 \quad (i=4,5,6) \tag{46}$$

When the expression for the rotational rigid body modes [Eq. (39)] is substituted into Eq. (46), we have

$$\int_D (y\phi_3^j - z\phi_2^j)\,\mathrm{d}m = 0 \tag{47a}$$

$$\int_D (z\phi_1^j - x\phi_3^j)\,\mathrm{d}m = 0 \tag{47b}$$

$$\int_D (x\phi_2^j - y\phi_1^j)\,\mathrm{d}m = 0 \tag{47c}$$

where the factor l inverse has been eliminated. If the three scalar equations are considered as components of the vector bases of the floating frame, then one vector cross-product relationship results from Eq. (46)

$$\int_D \bar{\rho} \times \phi^j \mathrm{d}m = 0 \tag{48}$$

where

$$\bar{\rho} = \{\hat{f}\}^T \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} \tag{49}$$

This vector cross-product expression [Eq. (48)] is recognized as the coefficient of the modal coordinate velocity $\dot{\eta}_j$ in the constraint relationship [Eq. (37)]. The constraint relationship can then be written without the deformational modal velocities and also without the translational rigid-body modal velocities. These latter are set to zero as a result of the center-of-mass constraint [Eq. (45)]. Writing the simplified expression for the constraint gives

$$\int_D \bar{\rho} \times \phi_4^0 \mathrm{d}m\dot{\eta}_{04} + \int_D \bar{\rho} \times \phi_5^0 \mathrm{d}m\dot{\eta}_{05} + \int_D \bar{\rho} \times \phi_6^0 \mathrm{d}m\dot{\eta}_{06} = 0 \tag{50}$$

Substituting the expression for the rigid-body modes into the foregoing yields

$$\{\hat{f}\}^T \left\{ \begin{bmatrix} \int_D (y^2+z^2)\mathrm{d}m \\ \int_D -yx\mathrm{d}m \\ \int_D -zx\mathrm{d}m \end{bmatrix} \dot{\eta}_{04} + \begin{bmatrix} \int_D -xy\mathrm{d}m \\ \int_D (x^2+z^2)\mathrm{d}m \\ \int_D -zy\mathrm{d}m \end{bmatrix} \dot{\eta}_{05} + \begin{bmatrix} \int_D -xz\mathrm{d}m \\ \int_D -yz\mathrm{d}m \\ \int_D (x^2+y^2)\mathrm{d}m \end{bmatrix} \dot{\eta}_{06} \right\} = 0 \tag{51}$$

The moments and products of inertia of the undeformed system are readily identified in the foregoing. Writing the resultant scalar equations in matrix form yields

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{Bmatrix} \dot{\eta}_{04} \\ \dot{\eta}_{05} \\ \dot{\eta}_{06} \end{Bmatrix} = 0 \qquad (52)$$

Since the inertia matrix is positive definite, the only solution of this constraint equation is

$$\dot{\eta}_{0i} = 0 \quad (i=4,5,6) \qquad (53)$$

This evaluation of the constraint relationship using free-free modes states definitively that the rigid-body mode frame is fixed relative to the Tisserand frame. If initially aligned, the two frames will remain coincident. Thus, the mode shape constraint, which involves a Tisserand frame, can be used to define a rigid-body mode frame. The result has also been presented recently by de Veubeke,[1] in work developed in parallel with that reported here.[9]

The procedure that results from applying the mode shape constraint is to expand the deformations in terms of the deformational modes only, ignoring the rigid-body modes, which have zero natural frequency. The constraint relationship is fulfilled because each coefficient in Eq. (31) is set to zero. The coordinates are then independent because of the specific choice of mode shapes. This interpretation gives rise to the name "mode shape constraint." The mode shape constraint may also be viewed as a straightforward way of working with the rigid-body mode frame. As noted before, the constraint relationship requires that the rigid-body modes do not move relative to the Tisserand frame. Thus, the mode shape constraint may be used to locate both frames. This is a much easier method to locate the rigid-body mode frame than a requirement involving zero strain energy.

The discussion will now take up the example problem. This is a case of planar motion of a long slender member comprised of two rigid links capable of small relative rotation at their connecting hinge; the constraint relationship will simplify to the scalar form

$$\int_D x\dot{u}\, \mathrm{d}m = 0 \qquad (54a)$$

If the velocity is expanded in terms of mode shapes, then

$$\sum_{j=0}^{n} \left(\int_D x\phi^j \mathrm{d}m \right) \dot{\eta}_j = 0 \qquad (54b)$$

The mode shapes for the system shown in Figs. 1 and 2 are the rigid-body mode, and one deformational mode relative to the Tisserand frame. They are shown in Fig. 4. The mode shapes are

$$\phi^0 = x/l \quad (-l/\sqrt{2} \le x \le l) \qquad (54c)$$

$$\phi^1 = -l/4 - 0.586x \quad (-l/\sqrt{2} \le x \le 0)$$

$$= -l/4 + 0.414x \quad (0 < x \le l) \qquad (54d)$$

Several characteristics of the mode shapes should be mentioned. They represent only small deformations and always place the origin at the center of mass [Eq. (32c)]. The deformational mode resulted from setting the relative deformation α to one. The rotation of the body-fixed frame relative to the Tisserand frame β would then be 0.586. The resultant movement of the center of mass would be $-l/4$. The essential property of these two modes is that they are orthogonal.

$$\int_D \phi^0 \phi^1 \mathrm{d}m = \int_D \frac{x}{l} \phi^1 \mathrm{d}m = 0 \qquad (54e)$$

Thus, the mass matrix is diagonal. Since the relative deformations of the system result from only the second mode shape, the stiffness matrix is zero except for k_{22}. The modes chosen, therefore, are the normal modes of the unconstrained system which we call the free-free modes. The system is unconstrained as the singular stiffness matrix would indicate.

Now let us use the free-free modes in order to evaluate the coefficients of the Tisserand constraint relationship shown in Eq. (54b).

$$\int_D x\phi^0 \mathrm{d}m = \int_{-l/\sqrt{2}}^{0} 2m\frac{x^2}{l}\,\mathrm{d}x + \int_0^l m\frac{x^2}{l}\,\mathrm{d}x = 2.218ml^2 \qquad (54f)$$

$$\int_D x\phi^1 \mathrm{d}m = 0 \qquad (54g)$$

The last coefficient is set to zero because of the orthogonality condition. The Tisserand constraint for this system may be written

$$\left(\int_D x\phi^0 \mathrm{d}m\right)\dot{\eta}_0 + \left(\int_D x\phi^1 \mathrm{d}m\right)\dot{\eta}_1 = 0 \qquad (2.218ml^2)\dot{\eta}_0 = 0 \qquad (54h)$$

The constraint requires that the rigid-body modal velocity be zero. Thus, the rigid-body mode does not move relative to the Tisserand frame. Since the original choice of axes is arbitrary, the modal coordinate may be set to zero and will remain zero because of the constraint.

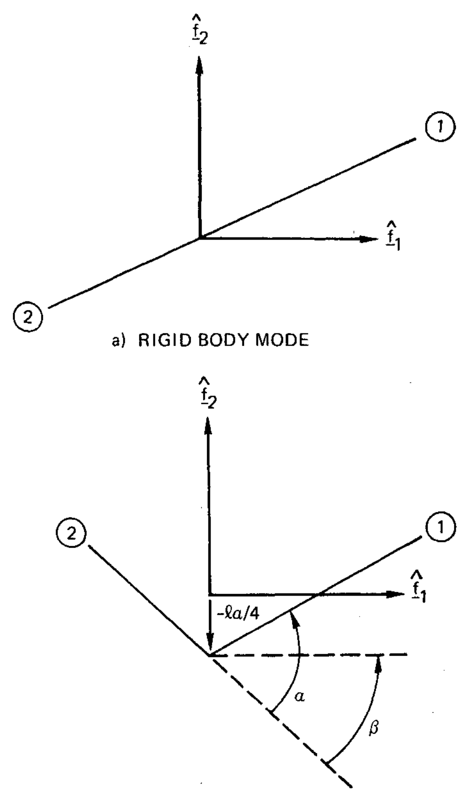

Fig. 4 Example problem mode shapes.

Modified Tisserand Constraint

This section will deal with an extension of the Tisserand frame for deformable bodies to cover a new class of systems.

Consider a deformable body with small relative displacements to which is added a spinning rigid rotor. The new system will no longer be a strictly deformable body, since the movement of the rotor will involve large relative displacements. If the classical Tisserand constraint is applied to this system, the frame itself must rotate relative to the system. The system angular momentum will consist of a component from the deformable body and a component from the rotor. The frame must rotate in such a way as to carry the total system angular momentum and thereby set internal angular momentum to zero. The frame motion cannot coincide with the deformable body component of the system and large deformations result.

Two methods of modifying the Tisserand constraint present themselves. The first is to define a Tisserand frame for the deformable body only. This approach ignores the rotor altogether when the internal angular momentum relative to the deformable body center of mass is set to zero. The constraint relationship is then written

$$\int_D \bar{\rho}_C \times \mathring{u}\ \mathrm{d}m = 0 \tag{55}$$

where $\bar{\rho}_C$ is the undeformed location relative to the center of mass of the deformable body. This approach is not used because it does not achieve the greatest simplification.

The second method to be considered is to examine the expression for the system angular momentum. The constraint relationship can then be made to simplify the structure of the angular momentum as much as possible while still assuring small deformations.

Before we write the expression for the system angular momentum, let us define several reference frames:

i = inertially fixed reference frame
f = floating frame
b = frame fixed in the mass element at the rotor mounting point
v = frame fixed in the rotor

The angular momentum of the system about its center of mass is

$$H = \int_S \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\ \mathrm{d}m \tag{56}$$

where S represents integration over the entire system, both the rotor and the deformable body. Evaluating the angular momentum for the deformable body will give

$$\int_D \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} \mathrm{d}m = \mathbf{I}_D \cdot \boldsymbol{\omega}^{fi} + \int_D \boldsymbol{\rho} \times \mathring{u}\ \mathrm{d}m \tag{57}$$

where $\mathbf{I}_D$ is the deformable body dyadic about the system center of mass. For the rotor, static balance has been assumed. The similar expression integrated over the rotor R is

$$\int_R \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}}\ \mathrm{d}m = \mathbf{I}_R^S \cdot \boldsymbol{\omega}^{fi} + \mathbf{I}_R \cdot \boldsymbol{\omega}^{vf} + M_R \boldsymbol{\rho}_R \times \mathring{\boldsymbol{\rho}}_R \tag{58}$$

where

$\mathbf{I}_R^S$ = rotor inertia dyadic about the system center of mass
$\mathbf{I}_R$ = rotor inertia dyadic about its own center of mass
M_R = rotor mass
$\boldsymbol{\rho}_R$ = location of rotor center of mass relative to system center of mass

By combining these two quantities, the system angular momentum is found to be

$$H = \mathbf{I} \cdot \boldsymbol{\omega}^{fi} + \mathbf{I}_R \cdot \boldsymbol{\omega}^{vf} + \int_D \boldsymbol{\rho} \times \mathring{u}\ \mathrm{d}m + M_R \boldsymbol{\rho}_R \times \mathring{\boldsymbol{\rho}}_R \tag{59}$$

The total system inertia dyadic is given by

$$\mathbf{I} = \mathbf{I}_D + \mathbf{I}_R^S \tag{60}$$

The total angular momentum expression can be significantly simplified. In order to do this, we introduce a modified Tisserand constraint.

$$\int_D \boldsymbol{\rho} \times \mathring{u}\ \mathrm{d}m + M_R \boldsymbol{\rho}_R \times \mathring{\boldsymbol{\rho}}_R + \mathbf{I}_R \cdot \boldsymbol{\omega}^{bf} = 0 \tag{61}$$

Here we have made use of the chain rule for angular velocities

$$\boldsymbol{\omega}^{vf} = \boldsymbol{\omega}^{vb} + \boldsymbol{\omega}^{bf} \tag{62}$$

A form of this constraint was also introduced by Willems for a dual-spin satellite with bearing flexibility.[4] The system angular momentum then becomes

$$H = \mathbf{I} \cdot \boldsymbol{\omega}^{fi} + \mathbf{I}_R \cdot \boldsymbol{\omega}^{vb} \tag{63}$$

The foregoing has the same structure as a rigid body with an attached rotor. The essential difference is that the inertia dyadic is not constant and that the relative angular momentum of the rotor (the second term) is affected by the deformations. It is important to understand that rotations of the axes at the mounting point will reorient the rotor axes.

The internal angular momentum of the system relative to the frame is no longer zero

$$\int_S \boldsymbol{\rho} \times \mathring{u}\ \mathrm{d}m = \mathbf{I}_R \cdot \boldsymbol{\omega}^{vb} \tag{64}$$

This is a very simple result and may be verified by direct computation. Alternatively, a quick comparison of the angular momentum expression in Eq. (63) and the general angular momentum expression in Eq. (3) will give the same result. The one-term evaluation of the internal angular momentum will facilitate the evaluation of the equations of motion. In order to demonstrate this, let us evaluate the kinetic energy. We start with the general result given earlier [Eq. (6)], which is

$$T = \frac{1}{2} \int_S \mathring{\boldsymbol{\rho}} \cdot \mathring{\boldsymbol{\rho}}\ \mathrm{d}m + \frac{1}{2} \boldsymbol{\omega}^{fi} \cdot \mathbf{I} \cdot \boldsymbol{\omega}^{fi} + \boldsymbol{\omega}^{fi} \cdot \int_S \mathring{\boldsymbol{\rho}} \times \mathring{\boldsymbol{\rho}}\ \mathrm{d}m \tag{65}$$

The first term must be evaluated over the deformable body and the rotor.

$$\frac{1}{2} \int_D \mathring{\boldsymbol{\rho}} \cdot \mathring{\boldsymbol{\rho}}\ \mathrm{d}m = \frac{1}{2} \int_D \mathring{u} \times \mathring{u}\ \mathrm{d}m \tag{66a}$$

$$\frac{1}{2} \int_R \mathring{\boldsymbol{\rho}} \cdot \mathring{\boldsymbol{\rho}}\ \mathrm{d}m = \frac{1}{2} M_R \mathring{\boldsymbol{\rho}}_R \cdot \mathring{\boldsymbol{\rho}}_R + \frac{1}{2} \boldsymbol{\omega}^{vb} \cdot \mathbf{I}_R \cdot \boldsymbol{\omega}^{vb} + \boldsymbol{\omega}^{bf} \cdot \mathbf{I}_R \cdot \boldsymbol{\omega}^{vb} + \frac{1}{2} \boldsymbol{\omega}^{bf} \cdot \mathbf{I}_R \cdot \boldsymbol{\omega}^{bf} \tag{66b}$$

Making the substitutions indicated will yield the kinetic energy for a deformable body and attached rigid rotor with static balance when the modified Tisserand constraint is employed

$$T = \frac{1}{2} \int_D \mathring{u} \cdot \mathring{u}\ \mathrm{d}m + \frac{1}{2} \boldsymbol{\omega}^{fi} \cdot \mathbf{I} \cdot \boldsymbol{\omega}^{fi} + \boldsymbol{\omega}^{fi} \cdot \mathbf{I}_R \cdot \boldsymbol{\omega}^{vb} + \frac{1}{2} M_R \mathring{\boldsymbol{\rho}}_R \cdot \mathring{\boldsymbol{\rho}}_R$$

$$+\frac{1}{2}\omega^{vb}\cdot\mathbf{I}_R\cdot\omega^{vb}+\omega^{vb}\cdot\mathbf{I}_R\cdot\omega^{bf}$$

$$+\frac{1}{2}\omega^{bf}\cdot\mathbf{I}_R\cdot\omega^{bf} \tag{67}$$

Even with the simplification that comes from the modified Tisserand constraint, this system has a complicated structure for the kinetic energy.

The modified Tisserand frame is distinct from the frame created by applying the classical Tisserand constraint to the deformable body only [Eq. (55)]. In order to demonstrate this, the internal angular momentum of the deformable body relative to its own center of mass will be calculated for the modified Tisserand constraint. Starting with Eq. (64) we may write

$$\int_D \rho\times\mathring{\rho}\,\mathrm{d}m+\int_R \rho\times\mathring{\rho}\,\mathrm{d}m=\mathbf{I}_R\cdot\omega^{vb} \tag{68}$$

Each component of internal angular momentum may be evaluated by rewriting the expression relative to its own center of mass rather than the system center of mass.

$$\int_D \rho\times\mathring{\rho}\,\mathrm{d}m=\int_D \bar{\rho}_C\times\mathring{u}\,\mathrm{d}m+(M-M_R)r_C\times\mathring{r}_C \tag{69a}$$

$$\int_R \rho\times\mathring{\rho}\,\mathrm{d}m=\mathbf{I}_R\cdot\omega^{vb}+\mathbf{I}_R\cdot\omega^{bf}+M_R\rho_C\times\mathring{\rho}_R \tag{69b}$$

where r_C is the location of deformable body center of mass relative to the system center of mass. By using the center of mass expression

$$(M-M_R)r_C+M_R\rho_R=0 \tag{70}$$

we may evaluate the internal angular momentum of the deformable body relative to its own center of mass

$$\int_D \bar{\rho}_C\times\mathring{u}\,\mathrm{d}m=-\mathbf{I}_R\cdot\omega^{bf}-\left(\frac{M_RM}{M-M_R}\right)\rho_R\times\mathring{\rho}_R \tag{71}$$

This quantity will not generally be zero, and the deformable body will have a net internal angular momentum relative to its own center of mass when the modified Tisserand constraint is employed. This result also shows that the modified Tisserand constraint involves an extension of the classical Tisserand constraint for deformable bodies.

Going back to total system angular momentum [Eq. (63)], it will be interesting to examine the frame motion. For the case where the system has zero net angular momentum and is free of external moments, the relationship will hold whereby

$$\mathbf{I}\cdot\omega^{fi}+\mathbf{I}_R\cdot\omega^{vb}=0 \tag{72}$$

This relationship does not require that the frame be inertially fixed. Instead, the frame will move when the internal angular momentum (the second term) is altered in value. This is the process of momentum exchange that is employed by control systems.

We shall now attempt to develop the most important feature of the modified Tisserand constraint, tying together the advantages of the mode shape constraint with the extended systems covered by the modified Tisserand constraint. The crux of this endeavor lies with the physical interpretation of the modified Tisserand constraint [Eq. (61)]. Let us start by discussing the terms present in the expression. The first term is the internal angular momentum of the deformable body. The last two terms involve the rotor, but neither term involves the rotor spin rate. Rather, these two terms represent the internal angular momentum of the rotor, if it were, indeed, not spinning relative to the axes fixed at the mounting point. That is, the rotor would be fixed in the reference frame b. The constraint relationship may now be interpreted as a classical linearized Tisserand constraint for a system with a "locked" rotor. Once this interpretation has been made, the mode shape constraint associated with a classical Tisserand frame may be applied to a deformable body with an attached rotor. The mode shapes used to expand the deformations are found from the free-free analysis of the system where the rotors have been locked; thus, the rotor mass and inertia properties contribute to the free-free modes. The modified Tisserand constraint will now require only that the rigid body modal velocities be zero. This allows them to be ignored, and the analysis proceeds with the frame variables and the modal coordinates associated with nonzero frequencies. All of these coordinates are independent, since the constraint relationship was evaluated by suppressing the rigid-body modes.

All of the relationships given earlier for a system with one rotor with static balance will hold if additional rotors are added. The only change is that expressions involving rotor quantities are summed for all rotors. Thus, the case of distributed actuators may be handled using the modified Tisserand constraint.

Summary

The advent of a new generation of larger spacecraft characterized by distributed structural flexibility provides renewed interest in floating reference frames whose motion is defined by constraint relationships. It is hoped that this paper will serve to identify the linearized Tisserand frame as the most advantageous choice. The qualities which recommend it are that it optimally follows the body; that the constraint relationship is easily evaluated; and that it can be extended to cover systems with momentum exchange controllers.

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